

Section E5C Impedance plots and coordinate systems: plotting impedances in polar coordinates; rectangular coordinates

Two Coordinate Systems

Rectangular Coordinates

We've been using rectangular coordinates, in which the location of any point is specified by two numbers, one number indicating how far the point is on a horizontal axis (the X axis) from some point taken as the origin, and another number indicating how far the point is on a vertical axis (the Y axis) from the origin. The origin is at coordinates 0,0.

Let's say our present location is an airport, and let's say we take the airport as the origin. We might instruct a pilot to fly to a point that's 40 miles east and 30 miles north-- in other words, at coordinates 40,30. (The coordinates are always stated in the order XY.)

Polar coordinates

That pilot, rather than flying 40 miles due west, making a 90-degree left turn, and flying 30 miles due north, would probably calculate that it would be more direct to fly 50 miles on a heading 36.9 degrees north of east. In other words, he'd convert it to *polar* coordinates, expressed as a distance and a bearing (angle). For our purposes,

- a line going to the right directly along the X axis is said to have an angle of 0 degrees
- a line going straight up along the Y axis has an angle of 90 degrees
- a line to the left along the X axis has an angle of 180 degrees
- a line straight down along the Y axis has an angle of 270 degrees

(A circle has 360 degrees. 270 degrees could also be called -90 degrees.)

The two numbers that are used to define a point on a graph using rectangular coordinates represent the coordinate values along the horizontal and vertical axes. (E5C11)

If you plot the impedance of a circuit using the rectangular coordinate system and find the impedance point falls on the right side of the graph on the horizontal line, what you know about the circuit is that it is equivalent to a pure resistance. (E5C12)

"The coordinate system that is often used to display the resistive, inductive, and/or capacitive reactance components of an impedance is rectangular coordinates." (E5C13)

The coordinate system that is often used to display the phase angle of a circuit containing resistance, inductive and/or capacitive reactance is polar coordinates. (E5C14)

"Trigonometry Lite"

As we've said before, vector resolution usually requires the use of "trig", which not all of us are trained in. But fortuitously, the questions on the exam confine themselves to a small repertoire of "standard" triangle types that can be committed to memory, bypassing the need for trig.

Trig, and our exam questions, are concerned with analyzing "right" triangles-- those in which one of the angles is a right (90-degree) angle.

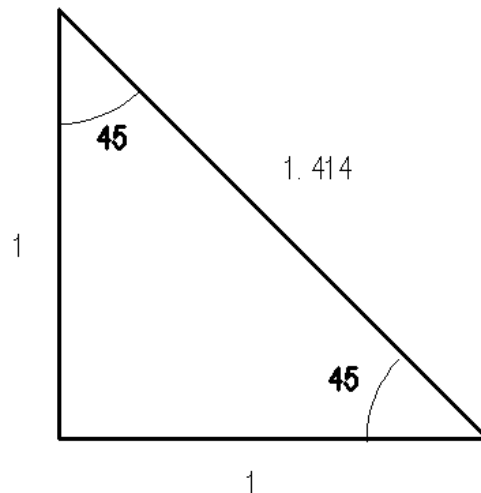
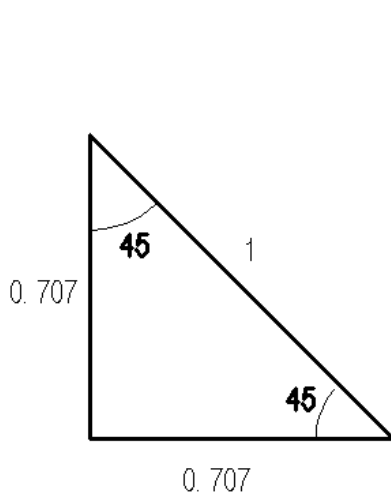
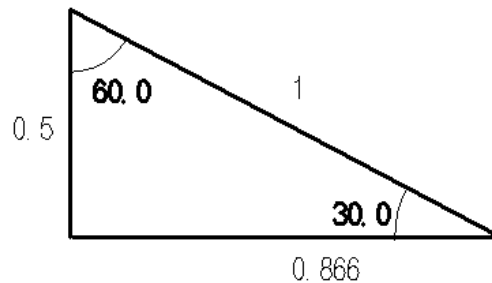
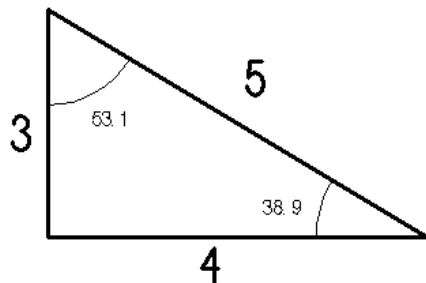
In most real-world cases, given the lengths of the two sides that meet at the right angle, the length of the hypotenuse (the third side) works out to be some oddball fractional number.

But in working the questions in the exam, certain manageable types of triangles keep turning up.

3-4-5 If the two sides have lengths of 3 and 4, the hypotenuse has a length of 5. (Applies also to triangles where the lengths are in the *ratio* of 3-4-5.)

30-60-90 If one of the angles in a right triangle is 30 degrees, another must be 60 degrees, and vice versa (since the three angles must add up to 180 degrees).

Isosceles (pronounced eye-SOSS-a-leez) triangles have 45-degree angles and two sides of equal length.



- In a 3-4-5 triangle, the two angles other than the right angle are 36.9 degrees and 53.1 degrees. (The three angles must add up to 180 degrees.)
- In a 30-60-90 triangle, the shorter of the sides has a length of one half the hypotenuse, and the longer has a length of 0.866 times the hypotenuse.
- In an isosceles triangle, the two angles are each 45 degrees. If the two sides have length = 1, the hypotenuse has a length of 1.414. If the hypotenuse has a length of 1, the two sides each have lengths of 0.707.

Now we can attack the questions (without knowing trig!).

Series questions

$R = 100$ ohms, $X_L = 100$ ohms, in series. Applying vector resolution we learned in Section E5B we know the vector trail ends at rectangular coordinates 100,100. We draw in the impedance vector, and recognize that we've drawn an isosceles triangle-- and so the polar coordinates must be 141 at 45 degrees. (E5C01)

A 100-ohm-reactance inductor, a 100-ohm-reactance capacitor, and a 100-ohm resistor in series. We see that the inductive reactance and capacitive reactance cancel out, so the vector trail ends right on the X axis at 100,0. So the impedance in polar coordinates is 100 ohms at an angle of 0 degrees. (E5C02)

A 300-ohm-reactance capacitor, a 600-ohm-reactance inductor, and a 400-ohm resistor in series. The vector trail ends at 400,300. Draw in the impedance vector and you see that you've completed a 3-4-5 triangle. So the polar coordinate must be 500 ohms at an angle of 37 degrees (an approximation of 36.9) . (E5C03)

A 400-ohm-reactance capacitor in series with a 300-ohm resistor. The vector trail ends at 300,-400. Draw in the impedance vector and you recognize a 3-4-5 triangle. The angle is toward 270 degrees, or toward minus 90 degrees. So the polar coordinate is thus 500 ohms at an angle of -53.1 degrees. (E5C04)

A 100-ohm reactance capacitor in series with a 100-ohm resistor. The vector trail ends at 100,-100 (down in the quadrant where the angles are negative). We recognize an isosceles triangle, so the polar coordinate must be 141 ohms at an angle of -45 degrees. (E5C06)

A 300-ohmreactance inductor in series with a 400-ohm resistor. The vector trail ends at 400,300. Draw the impedance vector and you recognize a 3-4-5 triangle. So the polar impedance is 500 ohms at an angle of 37 degrees. (E5C08)

Parallel questions

In the series circuits, we've plotted the resistance or reactance values. Implicitly, we were really plotting representations of the voltage values across each element-- since the current is inherently the same through all elements, each element voltage is proportional to the resistance or reactance values.

Parallel circuits call for a different approach. In a parallel circuit, all legs are inherently subject to the identical voltage, so we'll plot representations of the *currents* through the legs.
(Current is proportional to $1/Z$).

"E5C05 In polar coordinates, what is the impedance of a network consisting of a 400-ohm reactance inductor in parallel with a 300-ohm resistor?"

- A. 240 ohms at an angle of 36.9 degrees
- B. 240 ohms at an angle of -36.9 degrees
- C. 500 ohms at an angle of 53.1 degrees
- D. 500 ohms at an angle of -53.1 degrees

A rectangular plot *in terms of current* has a vector with a relative length of $1/300$ (the reciprocal of R , $1/R$) or 0.0033 along the X axis, then a vector straight up with a length of $1/400$ ($1/X_L$) or 0.0025.

(Don't fall into the error of thinking the impedance must be 500 ohms because of 3-4-5 triangle considerations with the 300-ohm and 400-ohm values-- we're working with the *reciprocals* of those values.) The length of the hypotenuse (the polar current vector) can be determined by geometry (Pythagoras' Theorem) as 0.00416.

Alternatively, we might spot that this is a 3-4-5 triangle:

$$0.0025 / 3 = 0.000833$$

$$0.000833 \times 4 = 0.0033$$

$$\text{So the hypotenuse} = 0.000833 \times 5 = 0.00416$$

The impedance of the circuit is thus $1/0.00416 = 240$ ohms at 36.9 degrees. Further, the multiple-choice answers presented allow the right one to be selected by an empirical, common-sense approach-- we know that when impedances are connected *in parallel*, the resulting impedance must be *lower* than the lowest of the individual impedances. So answers C and D are out. And we know from ELI the ICEman that in this inductive circuit the voltage leads the current, so the angle must be positive, so the right answer must be A. 240 ohms at an angle of 36.9 degrees. (E5C05)

E5C07 "In polar coordinates, what is the impedance of a network comprising a 100-ohm reactance capacitor in parallel with a 100-ohm resistor?"

- A. 31 ohms at an angle of -15 degrees
- B. 51 ohms at an angle of -25 degrees
- C. 71 ohms at an angle of -45 degrees
- D. 91 ohms at an angle of -65 degrees"

Here again, the multiple-choice answers allow an easy selection by an empirical common-sense approach-- with two equal ohmic values, the vector triangle is isosceles, so the angle must be 45 degrees or -45 degrees, so the right answer must be C. 71 ohms at an angle of -45 degrees. (E5C07)

Mathematicians perplexed!

The early mathematicians were quite troubled the first time they encountered calculations where they had to take the square root of a negative number.

Because there **IS NO** square root of a negative number.

If you were to ask "What's the square root of 81?", you're asking, in effect, "What number, if multiplied by itself, yields a product of 81?". And the answer is 9. ($9 \times 9 = 81$)

But what if you were to ask "What's the square root of -81?"

It's not 9, for $9 \times 9 = +81$.

And it's not -9, for $(-9) \times (-9) = +81$. (A negative times a negative = a positive.)

Hmmmmmm.....

The best they could come up with (and we still use it) was to say that

$$\sqrt{-81} = \sqrt{-1 \times 81} = \sqrt{-1} \times \sqrt{81} = \sqrt{-1} \times 9$$

All of that makes sense except $\sqrt{-1}$. They called that an *imaginary number*. We haven't found anything better, so we still use it. We represent it with a lower-case "j". So when we see something like " $\sqrt{-81}$ " we call it "j9", a representation of $(\sqrt{-1} \times 9)$.

Imaginary numbers find homes for themselves in lots of places, including reactive power, since reactive power is not "real" power (it's not consumed, as is resistive power). For example, the impedance of a series circuit comprising a 100-ohm resistor and a 100-ohm capacitor is expressed as 100 -j100 ohms.

Real numbers (such as resistance values) are graphed along the X axis; imaginary numbers (such as reactance values) are graphed along the Y axis.)

Numbers that combine real and imaginary components are called *complex numbers*.

"In polar coordinates, what is the impedance of a circuit of 100 -j100 ohms impedance?" We recognize an isosceles triangle, so the impedance is 141 ohms at an angle of -45 degrees. (E5C15)

Inverse quantities

Just as

- The inverse of resistance is conductance (G)

and

- The inverse of reactance is susceptance (B)

similarly,

- The inverse of impedance is admittance (Y)

$Y = 1/Z$ and is measured in siemens (S). (And conversely, $Z = 1/Y$)

"In polar coordinates, what is the impedance of a circuit that has an admittance of 7.09 millisiemens at 45 degrees?"

$$Z = 1/Y = 1/7.09 \times 10^{-3} = 0.141 \times 10^3 = 141 \text{ ohms.}$$

So the impedance is 141 ohms at an angle of -45 degrees. (E5C16) (The angle changed from positive to negative by virtue of taking the inverse.)

“In rectangular coordinates, what is the impedance of a circuit that has an admittance of 5 millisiemens at -30 degrees?”

The presence of the 30-degree angle tips us off that this is a 30-60-90 triangle..

$$Z = 1/Y = 1/.005 = 200 \text{ ohms.}$$

$$R = 200 * 0.866 = 173 \text{ ohms,}$$

$$X = 200 * 0.5 = 100 \text{ ohms.}$$

So the complex impedance is $173 + j100$ ohms. (E5C17)

“In polar coordinates, what is the impedance of a series circuit consisting of a resistance of 4 ohms, an inductive reactance of 4 ohms, and a capacitive reactance of 1 ohm?”

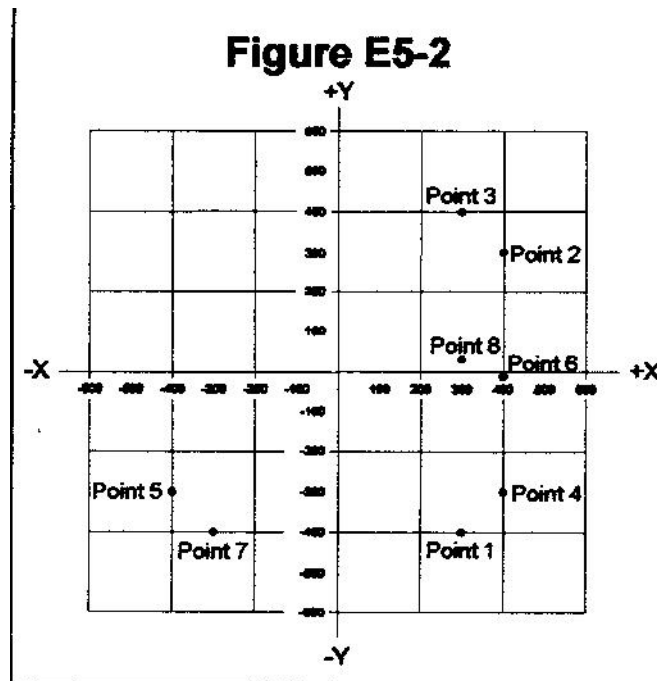
Either by drawing a vector diagram or by writing $4 + j4 - j1 = 4 + j3$, we recognize a 3-4-5 triangle. So the impedance is 5 ohms at an angle of 37 degrees. (E5C18)

“In rectangular coordinates, what is the impedance of a network comprising a 10-microhenry inductor in series with a 40-ohm resistor at 500 MHz?”

$$X_L = 2 * \pi * L = 6.28 * 10 * 10^{-6} * 500 * 10^6 = 31400 \text{ ohms.}$$

So the complex impedance is $40 + j31400$. (E5C22)

The remaining questions refer to Exam Fig. E5-2



“Which point on Figure E5-2 best represents the impedance of a series circuit consisting of a 400 ohm resistor and a 38 picofarad capacitor at 14 MHz?”

$$X_C = 1 / (2 * \pi * 38 * 10^{-12} * 14 * 10^6)$$

$$= 1 / 3341 * 10^{-6} = 290 \text{ ohms.}$$

400 - j290 is seen to correspond to **Point 4.** (E5C19)

“Which point in Figure E5-2 best represents the impedance of a series circuit consisting of a 300 ohm resistor and an 18 microhenry inductor at 3.505 MHz?”

$$X_L = 2 * 3.14 * 3.505 * 10^6 * 18 * 10^{-6} = 396 \text{ ohms}$$

300 + j396 is seen to correspond to **Point 3.** (E5C20)

“Which point on Figure E5-2 best represents the impedance of a series circuit consisting of a 300-ohm resistor, a 0.64-microhenry inductor and an 85-picofarad capacitor at 24.900 MHz?”

$$X_L = 2 * \pi * f * L = 2 * \pi * 24.9 * 10^6 * 0.64 * 10^{-6} = 100 \text{ ohms.}$$

$$X_C = 1 / 2 * \pi * 24.9 * 10^6 * 85 * 10^{-12} = 1 / 0.13292 = 75 \text{ ohms}$$

The polar impedance is thus $300 + j100 - j75 = 300 + j25$, point 8 (E5C23)

Prologue

Perhaps you're now wondering "In practical terms, what have I really learned? I haven't learned to analyze real-world reactive circuits, because in all probability I'm not going to encounter any of those 'standard' triangle types".

But you HAVE learned to analyze real-world circuits, because you've learned how to *graph* the problem, and then you're all set to determine the solution *off the graph* !

For example:

Suppose you have a series circuit with a 100 ohm resistor, a 25 uH inductor, and a 100 pf capacitor, and suppose you want to analyze it at a freq. of 3.75 MHz.

First, we'll have to determine the reactances at 3.75 MHz. (Of course, the reactances stated in any of the exam problems apply only at a particular freq.)

$$X_L = 2\pi fL = 2 * \pi * 3.75 * 10^6 * 25 * 10^{-6} = 589 \text{ ohms}$$

$$X_C = 1 / 2\pi fC = 1 / (2 * \pi * 3.75 * 10^6 * 100 * 10^{-12}) = 425 \text{ ohms}$$

So the net reactance is inductive, and equals $589 - 425 = 164$ ohms, and the complex impedance is $100 + j164$ ohms.

I drew that on graph paper, to a scale of 1 inch = 25 ohms. Then I drew in the hypotenuse with a ruler, measured its length (I got $7 \frac{11}{16}$ inches = 7.6875 inches) and scaled that up (multiplied it by 25) to yield 192.19 ohms. The precise answer obtained from Pythagoras' theorem, $\sqrt{100^2 + 164^2}$, is 192.08 ohms. Both are close approximations to 192 ohms.

I measured the phase angle with a protractor as 59 degrees. The precise angle obtained from trigonometry is 58.63 degrees.

So -- the graphical solution produces answers well within practical accuracy.